A Theorem of Tutte and 3D Mesh Parameterization

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Parameterization of 3D manifold mesh data involves embedding the mesh in some natural parametric domain, such as the plane or the sphere. Parameterization is important for many applications in geometry processing, including texture mapping, remeshing and morphing. The main objective is to generate a bijective mapping between the mesh surface and the parametric domain, which minimizes the distortion incurred in the transition in some meaningful sense. Examples of possible distortion are metric (edge length) distortion, conformal (angular) distortion and authalic (area) distortion.

A classical theorem of Tutte [7], originally designed to draw planar graphs, shows how to embed a manifold graph with the topology of a disk in the plane. This is achieved by fixing its boundary to a convex shape, and then solving a set of linear equations for the positions of the interior vertices. These equations express the fact that every interior vertex is positioned at the centroid of its neighbors. This basic method was later generalized by Floater [2] to arbitrary convex combinations, and Tutte's method could then be used to embed a 3D mesh in the plane, controlling the distortion by using convex weights derived from the geometry of the mesh. This class of embeddings are harmonic solutions of a discrete Laplace equation, namely requiring the weighted graph Laplacian operator to vanish at all interior vertices, with convex boundary conditions.

While Tutte's basic method remains a popular parameterization method, the constraint of a convex boundary is very severe, in most cases introducing unnecessary distortion into the result. Beyond that, it does not provide a satisfactory method to parameterize closed genus-0 meshes and meshes with higher genus. In this talk I will briefly survey some recent work of mine with colleagues on various generalizations of Tutte's method which overcome these problems.

Gortler, Gotsman and Thurston [3] provided conditions under which Tutte's method produces bijective embeddings even when the boundary is non-convex. This was used by Karni, Gotsman and Gortler [5] to generate free-boundary planar embeddings with constraints.

Gotsman, Gu and Sheffer [4] showed how to generalize the theory of Tutte to embed a closed genus-0 mesh on the sphere. This relies on recent algebraic characterizations of convex embeddings due to Colin de Verdiere [1], and related eigenvector constructions due to Lovasz and Schrijver [6]. In practice it involves solving a set of quadratic equations.

Inspired by recent work on discrete vector calculus, Gortler, Gotsman and Thurston [3] showed how the concept of a one-form from differential geometry can be defined on a discrete mesh. When such a one-form is harmonic, is may be used to generate bijective embeddings in the plane, in analogy to the Tutte method. The theory culminates in a discrete version of the Hopf-Poincare Index theorem, which may be used to provide a simple proof of the Tutte theorem, Moreover, it shows very simply (using a mere counting argument) how to generate a doubly-periodic embedding of the torus in the plane.

References

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