Folding Paper Shopping Bags

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1 Introduction. In grocery stores around the world, people fold and unfold countless paper bags every day. The rectangular-bottomed paper bags that we know today are manufactured in their 3D shape, then folded flat for shipping and storage, and later unfolded for use. This process was revolutionized by Margaret Knight (1838–1914), who designed a machine in 1867 for automatically gluing and folding rectangular-bottomed paper bags [8]. Before then, paper bags were cut, glued, and folded by hand. Knight's machine effectively demolished the working-class profession of "paper folder".

Our work questions whether paper bags can be truly (mathematically) folded and unfolded in the way that happens many times daily in reality. More precisely, we consider foldings that use a finite number of creases, between which the paper must stay rigid and flat, as if the paper were made of plastic or metal plates connected by hinges. Such foldings are sometimes called *rigid origami*, being more restrictive than general origami foldings, which allow continuous bending and curving of the paper and thus effectively uncountably infinite "creasing". It is known that essentially everything can be folded by a continuous origami folding [6], but that this is not the case for rigid origami.

We prove that the rectangular-bottomed paper bag cannot be folded flat or unfolded from its flat state using the usual set of creases that are so common in reality—in fact, the bag cannot move at all from either its folded or unfolded state. However, we show that a different creasing of a paper bag enables it to fold flat from its 3D state. We also conjecture a way to unfold a paper bag from its flat state if it was already folded using the usual set of creases (by an adversary equipped with techniques from origami or reality).

2 Related Work. In the mathematical literature, the closest work to rigid folding is *rigidity*. The famous Bellows Theorem of Connelly, Sabitov, and Walz [4] says that any polyhedral piece of paper forming a closed surface preserves its volume when folded according to a finite number of creases. In contrast, as suggested by the existence of bellows in the real world, it is possible to change the volume using origami folding. Even more fundamental are Cauchy's rigidity theorem, Aleksandrov's extension, and Connelly's extension [2], which all establish an inability to fold a convex polyhedron using a finite number of creases. (In Cauchy's case, the creases must be precisely the edges of the polyhedron; in Connelly's case, any finite set of additional creases can be placed; Aleksandrov's theorem is somewhere in between.) Another result of Connelly[1] is that a positive-curvature "corner" (the cycle of facets surrounding a vertex in a convex polyhedron) cannot be turned "inside-out" no matter how we place finitely many additional creases; this result answers a problem of Gardner [7]. In contrast, a paper bag can be turned inside-out with an origami folding (and in real life) [3].

Few papers discuss rigid origami directly. Demaine and Demaine [5] present a family of origami "bases" that can be folded rigidly. Streinu and Whiteley [9] proved that any single-vertex crease pattern can be folded rigidly—up to but not included the moment at which multiple layers of paper coincide. Balkcom and Mason [1] demonstrate how some classes of origami can be rigidly folded by a robot.

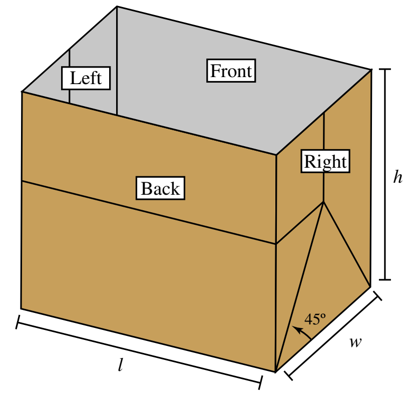

Figure 1: A shopping bag with creases in the usual places.

3 Main Results. Figure 1 shows a shopping bag with the usual crease pattern, and dimensions w, l, and h. For the bag shown, $h > w/2$, and $l > w$.

Our first main result states that a shopping bag cannot be folded at all with just the usual creases:

Theorem 1 *A shopping bag with the usual crease pattern has a configuration space consisting of two isolated points, corresponding to the fully-open and fully-closed configurations.*

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[1] Personal communication, 1998.

The results described in the previous section have two immediate consequences if we allow finitely many additional creases. First, the Bellows Theorem implies that, if the shopping bag had a top, no finite number of additional creases would allow the volume of the bag to be changed. Second, because the corners of the bag are convex, no finite number of additional creases would allow the shopping bag to be turned inside-out.

Based on these consequences, it might seem that no finite set of additional creases would allow a shopping bag to be folded flat. Our second result shows the opposite. A *short* shopping bag, with $h \leq w/2$, cannot have the usual shopping bag crease pattern, because the 45° creases do not intersect on the interior of the left and right sides of the bag; see Figure 2. In this case, we show

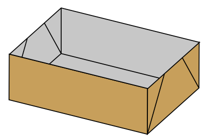

Figure 2: A short paper bag, similar to a collapsible department-store gift box.

Theorem 2 *Every short shopping bag (with $h \leq w/2$) can be collapsed flat using the creases in Figure 2.*

Now Theorem 2 suggests a method for folding a tall shopping bag: add creases to allow the tall bag to be telescoped until it is short enough to collapse flat. Figure 3 shows an animation of our procedure for shortening a bag by reducing h up to $\min\{w, l\}$. Using a sequence of these operations, we show

Theorem 3 *A tall shopping bag can be collapsed flat with the addition of finitely many creases.*

The collapsed state of the shopping bag after applying the folding technique described in the proof of theorem 3 is not the same as the collapsed state of the shopping bag with no additional creases. This difference suggests a more difficult question: can a collapsed shopping bag be opened up with the addition of a finite set of creases? We conjecture that it can, and propose a possible crease pattern in Figure 4.

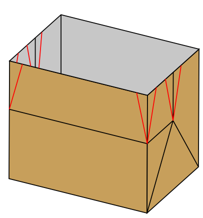

Figure 4: Conjectured creases for unfolding an already folded paper bag.

Conjecture 1 *A collapsed tall shopping bag can be unfolded with the addtion of a finite number of creases.*

If true, this conjecture would also offer a simpler way to flatten a tall shopping bag.

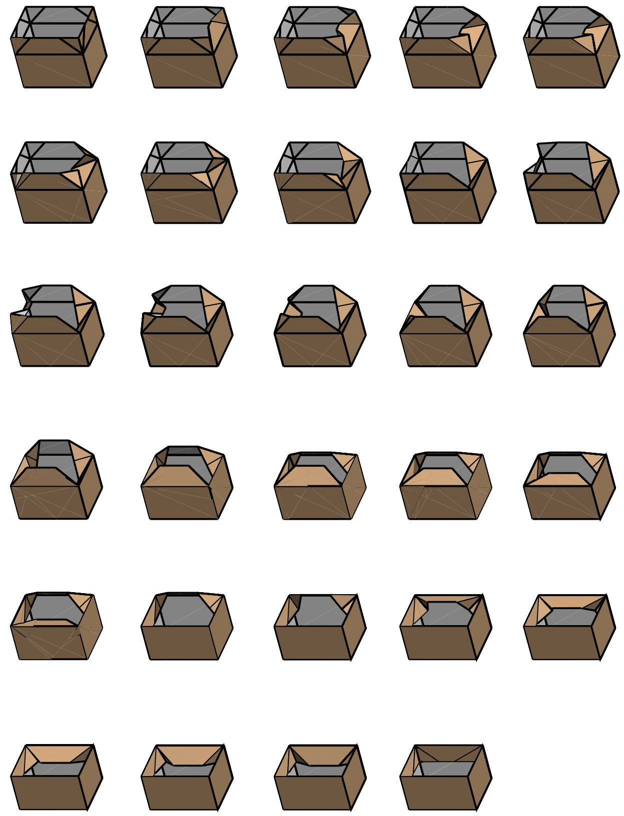

Figure 3: Procedure for shortening a rectangular tube.

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