A 2-chain Can Interlock with a k-chain

ABSTRACT

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Abstract

One of the open problems posed in [3] is: what is the minimal number k such that an open, flexible k-chain can interlock with a flexible 2-chain? In this paper, we establish the assumption behind this problem, that there is indeed some k that achieves interlocking. We prove that a flexible 2-chain can interlock with a flexible, open 16-chain.

1 Introduction

A *polygonal chain* (or just *chain*) is a linkage of rigid bars (line segments, edges) connected at their endpoints (joints, vertices), which forms a simple path (an *open chain*) or a simple cycle (a *closed chain*). A *folding* of a chain is any reconfiguration obtained by moving the vertices so that the lengths of edges are preserved and the edges do not intersect or pass through one another. The vertices act as universal joints, so these are *flexible chains*. If a collection of chains cannot be separated by foldings, the chains are said to be *interlocked*.

Interlocking of polygonal chains was studied in [4, 3], establishing a number of results regarding which collection of chains can and cannot interlock. One of the open problems posed in [3] asked for the minimal k such that a flexible open k-chain can interlock with a flexible 2-chain. An unmentioned assumption behind this open problem is that there is some k that achieves interlocking. It is this question we address here, showing that $k = 16$ suffices.

It was conjectured in [3] that the minimal k satisfies $6 \leq k \leq 11$. This conjecture was based on a construction of an 11-chain that likely does interlock with a 2-chain. We employ some ideas from this construction in the example described here, but for a 16-chain. Our main contribution is a proof that $k = 16$ suffices. It appears that using more bars makes it easier to obtain a formal proof of interlockedness.[1]

Results from [3] include:

1. Two open 3-chains cannot interlock.
2. No collection of 2-chains can interlock.
3. A flexible open 3-chain can interlock with a flexible open 4-chain.

This third result is crucial to the construction we present, which establishes our main theorem, that a 2-chain can interlock a 16-chain (Theorem 1 below.)

2 Idea of Proof

We first sketch the main idea of the proof. If we could build a rigid trapezoid with small rings at its four vertices (T_1, T_2, T_3, T_4), this could interlock with a 2-chain, as illustrated in Figure 1(a). For then pulling vertex v of the 2-chain away from the trapezoid would necessarily diminish the half apex angle α, and pushing v down toward the trapezoid would increase α. But the only slack provided for α is that determined by the diameter of the rings. We make as our subgoal, then, building such a trapezoid.

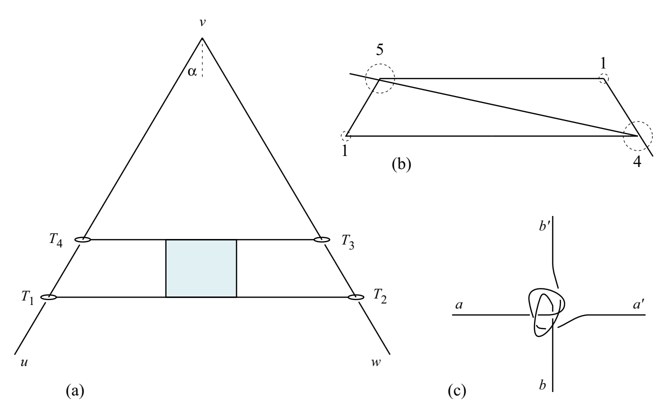

Figure 1: (a) A rigid trapezoid with rings would interlock with a 2-chain; (b) An open chain that simulates a rigid trapezoid; (b) Fixing a crossing of aa' with bb'.

We can construct a trapezoid with four links, and rigidify it with two crossing diagonal links. In fact, only

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[1] See http://arxiv.org/abs/cs.CG/0410052 for the full paper.

one diagonal is necessary to rigidify a trapezoid in the plane, but clearly a single diagonal leaves the freedom to fold along that diagonal in 3D. This freedom will be removed by the interlocked 2-chain, however, so a single diagonal suffices. To create this rigidified trapezoid with a single open chain, we need to employ 5 links, as shown in Figure 1(b). But this will only be rigid if the links that meet at the two vertices incident to the diagonal are truly "pinned" there. In general we want to take one subchain aa' and pin its crossing with another subchain bb' to some small region of space. See Figure 1(c) for the idea.

This pinning can be achieved by the "3/4-tangle" interlocking from [3], result (3) above; see Figure 2.

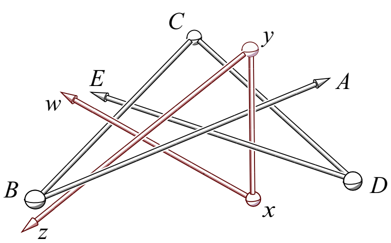

Figure 2: Fig. 6 from [3].

So the idea is replace the two critical crossings with a small copy of this configuration. This can be accomplished with 7 links per 3/4-tangle, but sharing with the incident incoming and outgoing trapezoid links potentially reduces the number of links needed per tangle. We have achieved 5 links at one tangle and 4 at the other. The other two vertices of the trapezoid need to simulate the rings in Figure 1(a), and this can be accomplished with one extra link per vertex. Together with the 5 links for the main trapezoid skeleton, we employ a total of $5 + (5 + 4 + 1 + 1) = 16$ links.

The final construction, shown in Figure 3, establishes our main result:

Theorem 1 *The 2-link chain is interlocked with the 16-link trapezoid chain.*

3 Discussion

We do not believe that $k = 16$ is minimal. We have designed two different 11-chains both of which appear to interlock with a 2-chain. However, both are based on a triangular skeleton rather than on a trapezoidal skeleton, and place the apex v of the 2-chain close to the 11-chain. It seems it will require a different proof technique to establish interlocking, for the simplicity of the proof presented here relies on the vertices of the 2-chain remaining far from the entangling chain.

Another direction to explore is closed chains, for which it is reasonable to expect fewer links. Replacing the 3/4-tangles with "knitting needles" configurations [2][1] produces a closed chain that appears interlocked, but we have not determined the minimum number of links that can achieve this.

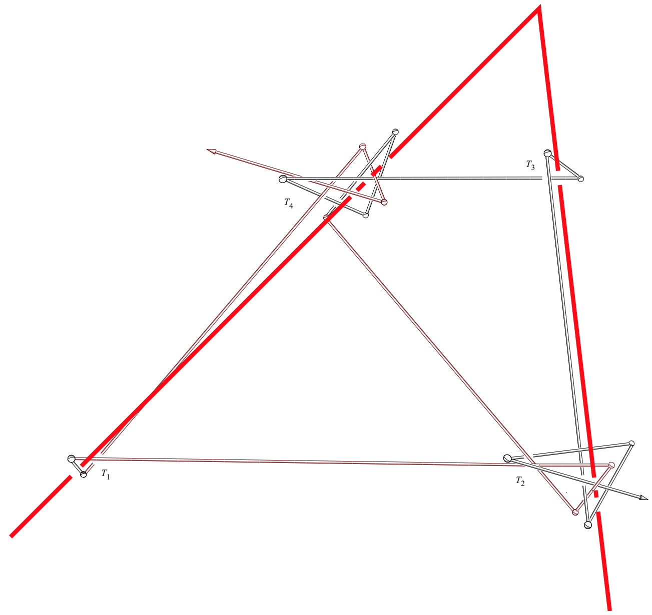

Figure 3: An open 16-chain forming a nearly rigid trapezoid, interlocked with a 2-chain (red).

Acknowledgements

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