Hamiltonian Cycles in Sparse Vertex-Adjacency Duals

Perouz Taslakian and Godfried Toussaint
School of Computer Science
McGill University
Montréal, Québec, Canada
{perouz, godfried}@cs.mcgill.ca

1 Introduction

In computer graphics, a common way of representing objects is with their triangulations. The performance of operations executed on these objects depends highly on how their surfaces are triangulated and how these triangles are transmitted to the processing engine. Thus, to speed up many operations, such as rendering or compression [9, 10], it is desirable that triangles be arranged such that their adjacency information is preserved.

In this paper we present a *sparse vertex-adjacency dual* of a polygon triangulation, which is a graph that preserves the vertex-adjacency information of the triangles and contains a Hamiltonian cycle. The size of this graph is linear in the number of polygonal vertices.

Effort has been made to design algorithms that produce Hamiltonian triangulations, where the dual graph of the triangulation is a path. In [1], Arkin et al. show that any set of n points has a Hamiltonian triangulation and describe two algorithms which construct such triangulations. They also show that the problem of determining whether a polygon (with holes) has a Hamiltonian triangulation is NP-complete. In the same paper, a *sequential triangulation of a set of points* is defined to be a Hamiltonian triangulation whose dual graph contains a Hamiltonian path, and it is proved that such triangulations do not always exist for any given set of points.

Hamiltonian properties of general triangulations have been studied extensively. Various results that construct a Hamiltonian cycle in a given triangulation can be classified based on the model considered. In one model, the given triangulation is allowed to be modified by adding new vertices or *Steiner* points. In [7], Gopi and Eppstein present an algorithm for constructing a Hamiltonian cycle in a given triangulation by inserting new vertices within existing triangles.

In the second model, the input triangulation cannot be modified. In this case, the problem is that of arranging adjacent triangles in some order such that the resulting graph contains a Hamiltonian cycle. An important property here is how *adjacency* is defined. In the dual graph of a triangulation, adjacency is defined as *edge-adjacency* where two triangles are adjacent when they share an edge. Unfortunately, it is not always possible to find Hamiltonian cycles in the dual graph.

Hamiltonian cycles in triangulations are studied when adjacency is defined as *vertex-adjacency*, where two triangles are considered to be adjacent if they share at least one vertex. In [5], a triangulation is represented with a *vertex-facet incidence graph* which has a vertex f for each facet (triangle), a vertex v for each triangle-vertex and an edge (v, f) whenever v is a vertex of triangle f. A *facet cycle* is defined by a walk $(v_0, f_1, v_1, f_2, v_2, \ldots, f_k, v_k, f_0, v_0)$ where no arc is repeated and that includes each facet vertex exactly once, but may repeat triangle-vertices. The authors prove that any triangulation has a facet cycle if it is not a *checkered* polygonal triangulation - that is if it does not have a 2-coloring of the triangles such that *every* white triangle is adjacent to three black ones.

A similar result under the same facet cycle model is found in [2]. Here, Bartholdi III and Goldsman refer to general triangulation as *Triangulated Irregular Networks (TINs)*. The authors describe an algorithm to construct a cycle in a 2-adjacent TIN (a triangulation in which each triangle shares an edge with at least two other triangles). Their algorithm runs in $O(n^2)$ time in the worst case.

In [4], Chen, Grigni and Papadimitriou define the *map graph* of a planar subdivision P (or a *map*) to be a graph G where the vertices of G correspond to the faces of P and two vertices u and v are adjacent if their corresponding faces in P share any point on their boundary. This characterization is equivalent to the dual graph of a triangulation in which two vertices u and v of the dual are connected by an edge whenever the triangles corresponding to u and v share a triangular edge or a triangular vertex. Chen et al. study sparsity and coloring of map graphs.

Bartholdi III and Goldsman [3] introduce the same concept of a map graph that they call the *vertex-adjacency dual* of a general triangulation. The authors show that the vertex-adjacency dual contains a Hamiltonian cycle, and they describe a linear time algorithm to construct such a cycle. Here we note that the model described in [3] is a variation of the facet-cycle model described in [5]: In the facet cycle model a continuous walk enters every triangle from a vertex v and leaves from a *different* vertex u. In the vertex-adjacency dual model, u and v are allowed to be the same vertex. The vertex-adjacency dual described in [3] can have $O(n^2)$ edges in the worst case. Here we consider linear size subgraphs of the vertex-adjacency dual that still contain Hamiltonian cycles, and that may be computed in linear time. We call such graphs *sparse vertex-adjacency duals*.

2 Constructing a Sparse Vertex-Adjacency Dual

Here we illustrate an approach with the simple case of sequential triangulations [6].

Let P be a simple polygon with n vertices and let T_P be a sequential triangulation of P. We will refer to the vertices of P as *polygonal vertices* and to the vertices of the dual graph D of T_P as *dual vertices*.

To construct the sparse vertex-adjacency dual of T_P, first construct its dual graph D, which in this case is a path. Then, for every polygonal vertex v_P, if v_P is shared by $k > 2$ consecutive triangles $t_1, t_2, \ldots, t_k$, insert an edge between the first and last triangles t_1 and t_k. The resulting graph G is a sparse vertex-adjacency dual. To show that it contains a Hamiltonian cycle, consider the following. In the dual graph D of T_P every consecutive vertices u, v and w are vertex-adjacent. Thus, connecting every u to w in the sparse vertex-adjacency dual is equivalent to connecting the vertices which are at distance 2 apart. The resulting graph is known as the *square* of D. In [8], it is shown that if removing the leaves of a tree T produces a path, then the square of T is Hamiltonian. In our case, our tree is the dual graph D, which will still be a path if we remove its leaves. Thus, from the results in [8] we can conclude that our construction for a sequential triangulation yields a Hamiltonian cycle (figure 1).

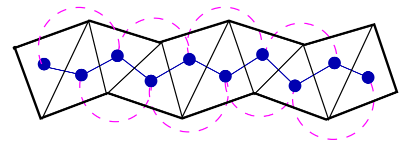

Figure 1: The sparse vertex-dual of a sequential triangulation is equivalent to the square D^2 of the dual graph D.

We can show that for any serpentine triangulation, the above construction will produce a graph that contains a Hamiltonian cycle. For general polygonal triangulations however, this construction needs a slight modification in order to contain such a cycle while preserving adjacency information of the triangles.

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