Grid Edge-Unfolding Orthostacks with Orthogonally Convex Slabs

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Abstract

We explore an instance of the question of unfolding an orthogonal polyhedron into one single connected piece that does not overlap itself and in which faces can only remain glued along whole edges.

1 Introduction

An *orthogonal* polygon P is a simple polygon with edges parallel to the coordinate axes. We use *horizontal* to refer to the x dimension and *vertical* to refer to the y dimension in the plane. P is called *orthogonally convex* if the intersection of P with any horizontal or vertical line is either empty or a single line segment.

A *slab* is a prism with an orthogonal polygon as basis and rectangular vertical faces. An *orthostack* S is an orthogonal polyhedron formed by stacking slabs $S_0, S_1, \ldots$ in the z dimension. Fig. 1 shows an orthostack example with two orthogonally convex slabs S_0 and S_1. If $z_0, z_1, z_2, \ldots$ are distinct

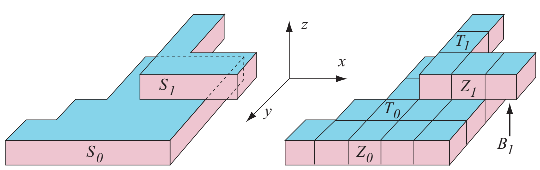

Figure 1: (a) An orthostack with two slabs S_0 and S_1. (b) Grid lines along which cuts are allowed.

z-coordinates of vertices of S, slab S_i is sandwiched between horizontal planes $z = z_i$ and $z = z_{i+1}$. The bottom B_i of S_i is the intersection of S_i with plane $z = z_i$, and the top T_i is the intersection of S_i with plane $z = z_{i+1}$. Note that T_i and B_i may be

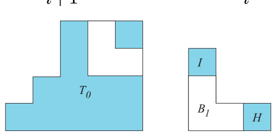

Figure 2: Top T_0 and bottom B_1 of the two slabs from Fig. 1; shaded pieces are exposed.

partly exposed (see the shaded pieces from Fig. 2) and partly interior to S.

In 3D we use *horizontal* to refer to a direction parallel to the xy plane and *vertical* to refer to a direction parallel to the xz or yz plane. The coordinate planes passing through every vertex of S induce a grid subdivision of S. We use the term *face* to refer to a surface rectangle in this subdivision. A *grid edge-unfolding* of S uses cuts along grid lines to unfold S into one connected planar piece in which faces remain connected along whole edges.

Demaine et. al[DIL04] show that all orthostacks can be *grid vertex-unfolded*, allowing faces to remain connected at single vertices. This abstract focuses on grid edge-unfolding of orthostacks with orthogonally convex slabs that satisfy an additional restriction: each maximal exposed piece of T_i contains two orthogonally incident edges of B_{i+1}.

A few more definitions are needed to present the algorithm. For any vertical face f of S, let $\alpha(f)$ be the space enclosed between the two planes orthogonal to f passing through the vertical edges of f. Define strip $s_i(f)$ as the maximum exposed connected component adjacent to f that lies in $\alpha(f) \cap (T_i \cup B_{i+1})$. For instance, the strip labeled C in Fig. 3b is precisely $s_1(L_0)$. The *band* (or *zone*) Z_i consists of all vertical faces of S_i. For any element x of S, we use the primed symbol x' to refer to x in the 2D unfolding.

2 Unfolding Algorithm

We proceed with an informal description of the unfolding with the help of the example from Fig. 1. Each band Z_i is cut along a vertical edge and unfolded in the plane horizontally, as in Fig. 3. Two vertical faces of each band Z_i are particularly important: $L_i^{'}$, the leftmost face and $R_i^{'}$, the rightmost face in the unfolding of Z_i. Bands for adjacent slabs are connected vertically by a *bridge*, a connected set of horizontal faces of S. A bridge connects $R_i^{'}$ to $L_{i+1}^{'}$ such that $L_{i+1}^{'}$ always lies vertically aligned or to the right of $R_i^{'}$ in the unfolding, as in Fig. 3. Exposed pieces of T_i and B_i are attached as vertical strips to opposite sides of $Z_i^{'}$.

2.1 Determining R_i, L_{i+1} and Bridge $R_i \rightarrow L_{i+1}$

Suppose that Z_{i-1}, L_i and the bridge connecting R_{i-1} to L_i have been flattened out in the plane.

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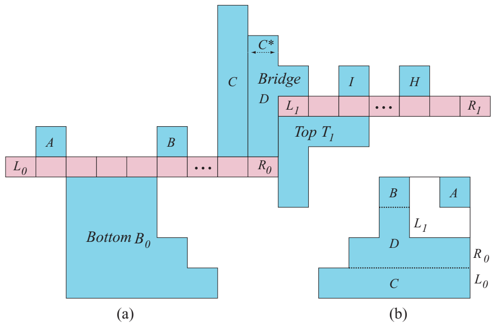

Figure 3: (a) An unfolding of S from Fig. 1 with bridge D. (b) Partition of T_0 into strips.

W.l.o.g., assume that the face of Z_i adjacent to the right edge of $L_i^{'}$ lies ccw from L_i, when viewed from $z = \infty$. For any face f of Z_i, let $\mathcal{H}(f)$ denote the open halfspace that contains f and is bounded by the vertical plane orthogonal to f passing through the vertical edge of f first encountered in a ccw walk from L_i.

Face R_i is not always clockwise adjacent to L_i. We pick R_i to be the face of Z_i last encountered in a ccw walk starting at L_i, with $R_i \neq L_i$, such that either R_i is adjacent to Z_{i+1}, or both of the following hold:

(1) The halfspace $\mathcal{H}(R_i)$ contains a face f of Z_{i+1} parallel to R_i

(2) There exists an exposed connected component $\mathcal{C}$ in $\mathcal{H}(R_i) \cap T_i$ or $\mathcal{H}(R_i) \cap B_{i+1}$ that is adjacent to both R_i and f.

We claim that R_i always exists. Furthermore, several faces f may satisfy conditions (1) and (2) above. We pick L_{i+1} to be the face f whose distance to the plane bounding $\mathcal{H}(R_i)$ is smallest. In case of a tie, L_{i+1} is the face *closest* to R_i, and thus uniquely defined.

Note that the subband $[L_i, R_i]$ ccw from L_i to R_i (when viewed from $z = \infty$) may not cover the entire Z_i (see Fig. 4b). However, we can show that $[L_i, R_i]$ is not too small either. In particular, $[L_i, R_i]$ contains at least two of the four monotone chains of the orthogonally convex polygon T_i. It is the existence of these chains that ensures that any face in $Z_i \setminus [L_i, R_i]$ can be attached by a vertical strip to one of the faces in $[L_i^{'}, R_i^{'}]$.

Having fixed R_i and L_{i+1}, we can now determine the bridge $R_i \rightarrow L_{i+1}$. If R_i is adjacent to Z_{i+1}, then R_i and L_{i+1} are adjacent and the bridge is empty. Otherwise, let $\mathcal{P}$ be the plane bounding the halfspace $\mathcal{H}(R_i)$ and let $\mathcal{C}$ be the maximum component that meets condition (2). The bridge $R_i \rightarrow L_{i+1}$ is $\mathcal{C}^* \cup s_i(L_{i+1})$, where $\mathcal{C}^*$ is the piece of $\mathcal{C}$ that lies between $\mathcal{P}$ and the plane parallel to $\mathcal{P}$ containing the vertical edge of L_{i+1} closer to $\mathcal{P}$.

2.2 Unfolding T_i, B_i and Remaining Z_i

Assume that the bridge $R_i \rightarrow L_{i+1}$ lies in $\mathcal{H}(R_i) \cap T_i$. To unfold T_i, we partition the exposed parts of T_i into strips as follows. First we delimit the region $\mathcal{X}$ of T_i enclosed between two planes parallel to L_i, one that contains L_i and the other passing through the vertical edge of R_i closer to L_i. Refer to Fig. 4b. We state without proof that $\mathcal{X}$ is entirely exposed. Next we partition $\mathcal{X}$ into strips parallel to L_i and $T_i \setminus \mathcal{X}$ into strips orthogonal to R_i, as in Figs. 3b and 4b. Each strip thus formed is precisely $s_i(f)$, for some f in $[L_i, R_i]$; we attach $s_i(f)$ above/below f', to complete the unfolding of T_i. We similarly unfold exposed faces of B_i (except for B_0, which we unfolded as one single piece).

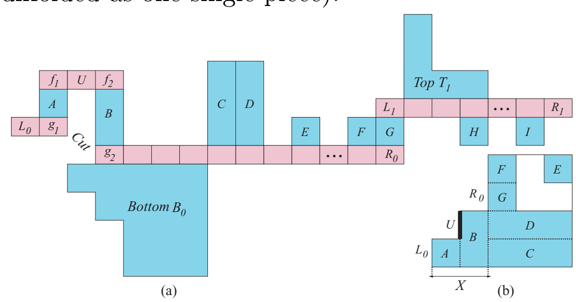

Figure 4: (a) An unfolding of S from Fig. 1 with bridge G. (b) Partition of T_0 into strips: A, B are parallel to L_0, and $C \ldots G$ are orthogonal to R_0.

If L_i and R_i are not adjacent in Z_i, the subband $Z_i \setminus [L_i, R_i]$ is yet to be unfolded. See Fig. 4b. We attach faces f in $Z_i \setminus [L_i, R_i]$ to vertical strips $s_i(f)$ already unfolded (see faces f_1 and f_2 from Fig. 4a). On top of each such face $f^{'}$ we attach vertically $s_{i-1}(f)$, if any. At the end of this process, all exposed faces of B_i have been unfolded.

Suppose now that two parallel faces f_1 and f_2 in $Z_i \setminus [L_i, R_i]$ are connected by a subband U of Z_i orthogonal to f_1 and f_2. Let g_1 and g_2 be the faces at the other end of the flattened strips $s_i(f_1)$ and $s_i(f_2)$. Then we cut $Z_i^{'}$ along the vertical edge shared by g_1 and g_2, and reconnect $s_i(f_1)$ and $s_i(f_2)$ by horizontal subband U. The case in which the bridge lies in $\mathcal{H}(R_i) \cap B_{i+1}$ is handled similarly.

References

[DIL04] E. D. Demaine and J. Iacono and S. Langerman. Grid Vertex-Unfolding of Orthostacks. In *Proc. of the Japan Conf. of Discr. and Comp. Geom.*, 2004.