Tightening: Curvature-Limiting Morphological Simplification

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Given a planar set S of arbitrary topology and a radius r, we define an r-tightening of S, which is a set that has a radius of curvature everywhere greater than or equal to r and that only differs from S in a morphologically-defined tolerance zone. This zone, which we call the mortar, contains only the details of S, such as high curvature portions of its boundary, thin gaps and constrictions, and small holes and connected components. We describe how to approximately compute r-tightenings for shapes represented as binary images using constrained, level set curvature flow.

Our work addresses a formulation of the shape smoothing problem different from those given in most prior art. The energy minimization-based fairing methods in the CAD/CAM literature (such as [3]) and the various polygon mesh smoothing techniques in the graphics literature (such as [2]) typically do not guarantee a bound on curvature or confine shape changes to a tolerance zone like the mortar.

The mortar, which we introduced in [7], is defined in terms of the morphological operations of rounding and filleting, which are described in detail in [4]. S rounded by r, denoted $R_r(S)$, is the union of disks of radius r contained in S, while S filleted by r, denoted $F_r(S)$, is the complement of the union of disks contained in the complement of S. The mortar is $F_r(S)-R_r(S)$. It is empty away from thin and high-curvature regions of S, and around those regions it is a subset of all points within a distance r of the boundary of S.

We define a simple closed curve C lying in a set T as tight with respect to T if it locally minimizes length, so that there exists a ε such that for all t and all $\delta \le \varepsilon$, $d(C(t),C(t+\delta)) = \delta$, where d(A,B) is the length of the shortest path connecting A and B in T and C is parameterized by arclength. We define a point on the boundary of a shape as concave if the line segment connecting the intersections of a small circle centered on the point with the boundary lies completely outside the shape. Tight loops through a set consist of concave portions of its boundary connected by tangent line segments. Because concave portions of the boundary of the mortar have a radius of curvature greater than or equal to r, if we define an r-tightening of S as a set T, $R_r(S) \subseteq T \subseteq F_r(S)$, such that the bounding loops of T are tight with respect to the mortar of S, it follows that the boundary of an r-tightening also has a radius of curvature greater than or equal to r.

When $R_r(S)$ and the complement of $F_r(S)$ each consist of a single connected component, the tightening is unique, and its boundary is the shortest loop around $R_r(S)$ lying in $F_r(S)$. In this case the tightening corresponds to the relative convex hull or minimum perimeter polygon [6] of $R_r(S)$ in $F_r(S)$. When $R_r(S)$ and $F_r(S)$ have more complex topologies, there may be several different tightenings, each of which may have holes and multiple connected components

We conjecture that for shapes of arbitrary topology represented as binary images, level-set curvature flow [5] constrained to the mortar always converges to a tightening, which includes as a corollary that a tightening always exists. In our implementation of curvature flow, we initialize the level set function Φ to be the signed Euclidean distance to the boundary of the core of the input shape, approximately computed using

Danielsson's vector propagation algorithm [1], which we also use for implementing the morphological operations. At each iteration, we compute $\Phi_t = -F\,|\nabla\Phi|$, where F is the velocity of the level set, which is equal to the curvature in the mortar and zero outside the mortar. The curvature is given by

$$\kappa = \nabla \cdot \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{\Phi_{xx}\Phi_y^2 - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_{yy}\Phi_x^2}{\left(\Phi_x^2 + \Phi_y^2\right)^{3/2}}$$

Where $|\nabla\Phi| = (\Phi_x^{\,2}+\Phi_y^{\,2})^{1/2}$, and the partial derivatives are computed using finite differences. We then compute $\Phi(t+\Delta t, x, y) = \Phi(t, x, y) + \Delta t \bullet \Phi_t(t, x, y)$, where Δt is inversely proportional to the maximum value of F at time t, so that the level set crosses at most one pixel each iteration. During most iterations, we only update Φ in a narrow band of pixels around the zero level set.

Curvature flow converges slowly where the radius of curvature spans several pixels. We therefore downsample the image representation of the core by a factor of two until r corresponds to 1-2 pixels. We perform the flow at this coarse resolution, then iteratively upsample by a factor of two and re-perform the flow. We find we need less than 100 iterations at each level of resolution. We anticipate adapting this technique to generate three-dimensional results using mean curvature flow.

References

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