Fast almost-linear-sized nets for boxes in the plane

Hervé Brönnimann*
Jonathan Lenchner
CIS, Polytechnic University, Six metrotech, Brooklyn, NY 11201, USA.

1 Introduction

Let $\mathcal{B}$ be any set of n axis-aligned boxes in $\mathbb{R}^d$, $d \geq 1$. For any point p, we define the subset $\mathcal{B}_p$ of $\mathcal{B}$ as $\mathcal{B}_p = \{B \in \mathcal{B} : p \in B\}$. A box B in $\mathcal{B}_p$ is said to be *stabbed* by p. A subset $\mathcal{N} \subseteq \mathcal{B}$ is a $(1/c)$-*net* for $\mathcal{B}$ if $\mathcal{N}_p \neq \emptyset$ for any $p \in \mathbb{R}^d$ such that $|\mathcal{B}_p| \leq n/c$. The number of distinct subsets $\mathcal{B}_p$ is $O((2n)^d)$, so the set system described above has so-called finite VC-dimension d. This ensures that there always exists $(1/c)$-nets of size $O(dc\log(dc))$, and that they can be found in time $O_d(n)c^d$, using quite general machinery (see for example the books by Matoušek [3] or by Pach and Agarwal [7]). For some set systems, such as halfplanes in $\mathbb{R}^2$ and translates of a simple closed polygon, it was shown that there exist $(1/c)$-nets of size $O(c)$ [4]. This was extended to halfspaces in $\mathbb{R}^3$ and pseudo-disks[1] in $\mathbb{R}^2$ [2].
In this paper, we investigate a fast, $O(n\log c)$-time construction of $(1/c)$-nets of size $O(c)$ for any value $1 < c \leq n$ and $d = 2$. Until right before JCDCG, I thought I could prove the following (which unfortunately remains a conjecture):

Conjecture 1 *Let $\mathcal{B}$ be a set of n axis-aligned boxes in $\mathbb{R}^2$ and c be any parameter $1 \leq c \leq n$. Then there exists a $(1/c)$-net $\mathcal{N}$ for $\mathcal{B}$ of size $O(c)$.*

We can prove this result and also provide algorithms that run in time $O(n\log c)$ only for special cases: segments on the real line (the one-dimensional case), quadrants of the form $(-\infty, x] \times (-\infty, y]$ in $\mathbb{R}^2$, and unbounded boxes of the form $[x_1, x_2] \times (-\infty, y]$ (which we call a *skyline*). For the general case of boxes, we can prove a size bound of $O(c\log\log c)$. But the conjecture still stands.

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[1]In this context, a collection of shapes is called a pseudo-disk set system if given any three points, there is at most one shape in the collection whose boundary passes through these three points.

2 Intervals on the line

We first prove that it is easy to find small nets for intervals on the line, the one-dimensional case of the problem above.

Theorem 2 *Let $\mathcal{B}$ be a set of n intervals on the real line $\mathbb{R}$ and c be any parameter $1 < c \leq n$. There exists a subset $\mathcal{N}$ of at most $2\lceil c-1\rceil$ boxes in $\mathcal{B}$ that is a $(1/c)$-net for $\mathcal{B}$. Such a set can be found in $O(n + n\log c)$ time.*

3 Rectangles in the plane

We generalize the method of the previous paragraph to the plane. We begin with the easier problem when all the boxes are south-west quadrants, i.e. they contain the point $(-\infty, -\infty)$.

Theorem 3 *Let $\mathcal{B}$ be a set of n quadrants with the same orientation in $\mathbb{R}^2$, and c be any parameter $1 < c \leq n$. Then there exists a $(1/c)$-net $\mathcal{N}$ for $\mathcal{B}$ of size $\lceil c-1\rceil$. Such a net can be found in time $O(n\log c)$.*

Let us add one more side to the quadrants: a *skyline* is a set of boxes that all intersect a common line. We are only interested in what happens on one side of that line, so we can consider unbounded boxes of the form $[x_1, x_2] \times (-\infty, y]$. We can extend the previous result to a skyline.

Theorem 4 *Let $\mathcal{B}$ be a set of n axis-aligned boxes, all unbounded in some common direction, and c be any parameter $1 < c \leq n$. Then there exists a $(1/c)$-net $\mathcal{N}$ for $\mathcal{B}$ of size at most $\lceil 2c-1\rceil$. Such a net can be found in time $O(n\log c)$.*

Using this result, we can now solve the general problem for boxes. Unfortunately, we cannot solve the conjecture, but we can prove:

Theorem 5 *Let $\mathcal{B}$ be a set of n axis-aligned boxes in $\mathbb{R}^2$ and c be any parameter $1 < c \leq n$. Then there exists a $(1/c)$-net $\mathcal{N}$ for $\mathcal{B}$ of size $O(c \log \log c)$. Such a net can be found in time $O(n \log c)$.*

4 k-oriented objects

A natural generalization of boxes in $\mathbb{R}^2$ is that of a k-oriented convex polygon [6], which is simply a convex polygon whose sides are constrained to be parallel to a set of k fixed directions ($k = 2$ for boxes). Our proof extends there as well. In fact, we suspect that any result for boxes would extend to k-oriented polygons where the constants in the $O()$ notations become functions of k. But we have no proof of such a general statement.

5 Orthants in higher dimension

As shown in the previous section, the key problem is that for the generalized orthants, which we call orthants for short. We are now interested in this problem for any dimension. We first prove that finding nets for orthants does indeed help for boxes.

Theorem 6 *Assume that there exists ε-nets of size $s(\varepsilon)$ for any set of orthants in $\mathbb{R}^d$, and that $s()$ is non-decreasing and has polynomial growth (which implies $s(O(x)) = O(s(x))$). Let $\mathcal{B}$ be a set of n boxes in $\mathbb{R}^d$ and c be any parameter $1 \leq c \leq n$. Then there exists a $(1/c)$-net $\mathcal{N}$ for $\mathcal{B}$ of size $O_d(s(1/c))$. Such a net can be found in time $O(n \log c)$.*

Now we show how small nets we can find efficiently for orthants. The bound is far from optimal for dimensions greater than 2, as nets of size $O_d(c \log c)$ exist but take much longer to compute (see the introduction).

Theorem 7 *Let $\mathcal{B}$ be a set of n orthants with the same orientation in $\mathbb{R}^d$ and c be any parameter $1 \leq c \leq n$. Then there exists a $(1/c)$-net $\mathcal{N}$ for $\mathcal{B}$ of size $O_d(c^{d-1})$. Such a net can be found in time $O(n \log c)$.*

For points and halfspaces in $\mathbb{R}^d$, Matoušek, Seidel, and Welzl [4] have shown that there exist ε-nets of size $O(1/\varepsilon)$. They also show that it suffices to restrict to points in convex position, albeit by having nets bigger by a factor of d. We prove an analog result for orthants, without the blowup factor. The analogue of convex position for orthants is maximal position, as defined in [8].

Lemma 8 *Suppose there exists a ε-net of size $s(\varepsilon)$ for any set of orthants in $\mathbb{R}^d$ in maximal position. Then there exists an ε-net for any set of orthants in $\mathbb{R}^d$ of size $s(\varepsilon)$.*

6 Conclusion

This shows another set system where the general bound $O(c \log c)$ for a $(1/c)$-net could be improved to $O(c)$, and more efficient algorithms can be found. Komlos, Pach and Woeginger [2] have shown that there exist set systems for which $(1/c)$-nets must have size $\Omega(c \log c)$.
This also poses the analog problem of finding good approximations, in the sense that not only does p hit few boxes if it misses $\mathcal{N}$, but the number of hits in $\mathcal{N}$ reflects the number of hits in $\mathcal{B}$ (scaled by $|\mathcal{N}|/|\mathcal{B}|$). The approach above seems to collapse because nothing guarantees the representativity of $\mathcal{N}$.

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