Drawing Equally-Spaced Curves between Two Points

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1 Problem Definition

Given two points a and b in the plane, draw n equally-spaced curves, $C_1, C_2, \ldots, C_n$ between them. More formally, those curves must satisfy the equi-distant property: for any point p on any such curve $C_i, 1 \leq i \leq n$ the two adjacent curves C_{i-1} and C_{i+1} are equally distant, that is, $d(p, C_{i-1}) = d(p, C_{i+1})$, where $d(p, C)$ is the distance from p to the point on a curve C that is closest to p. We define $C_0 = a$ and $C_{n+1} = b$ as degenerate curves. So, we have $d(p, C_0) = d(p, a)$ and $d(p, C_{n+1}) = d(p, b)$.

This problem is related to wire routing in printed circuit boards. In a new production technology more than one wires can be put between two pins. Then, it is desired to draw those wires so that they are equally spaced due to electrical constraints.

It is easy to draw one curve (line) between two points. It is simply a perpendicular bisector of the two points. It is also rather easy to draw three equally-spaced curves. The middle one is the perpendicular bisector of the two points and the remaining two curves are parabolas, each of which is a curve equidistant from a point and a line.

The case $n = 2$ is not easy. In this short paper we prove that we can draw two such equally spaced curves although we have no analytic equations for the curves.

2 Basic Properties of the Curves

For simplicity but without loss of generality we fix the two points a and b: $a = (3, 0)$ and $b = (-3, 0)$. We denote the two curves by C_a and C_b, which are characterized as follows:
(1) For any point p on C_a, $d(p, a) = d(p, C_b)$, and
(2) for any point p on C_b, $d(p, b) = d(p, C_a)$.

The curves C_a and C_b must pass through the points $(1, 0)$ and $(-1, 0)$, respectively, and the distance $d(p, C_a)$, for example, is given as the length of a line segment directed from p perpendicularly to the curve C_a.

Now, take any point p on C_a. Then, we have $d(p, a) = d(p, C_b)$ It implies that the circle centered at p with the point a on it must be tangent to the curve C_b at a unique point, which is denoted by q_p. The point q_p on C_b is called an image of the point p on C_a.

Fig. 1 shows this tangential property.

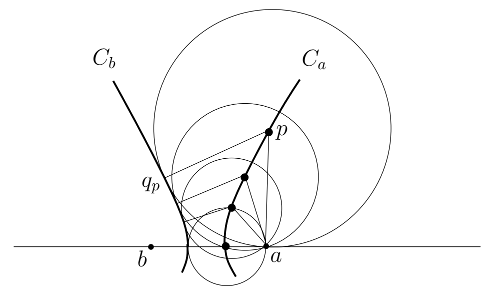

Figure 1: Tangential property.

By this tangential property we can guess a rough shapes of C_a and C_b, that is, they are smooth and convex toward the origin since they are envelopes of circles. As a point p goes to infinity along the curve C_a, the radius of its associated circle approaches to infinity, and finally it converges to a line.

3 Drawing Curves

The properties described above suggest equations specifying the curves C_a and C_b, but it seems to be hard to obtain analytic representations of the curves. Our strategy here is to compute rough shapes of the curves. For that purpose, we partition the plane into small squares of side length ε and remove all squares that cannot intersect the curves (see Fig. 2 for illustration). Due to the symmetry of the curves C_a and C_b, we only consider C_a.

Given a value of $\varepsilon > 0$, the plane is partitioned into squares. Each such square is specified by two-dimensional indices, as

$$s_{i,j} = [i\varepsilon, (i+1)\varepsilon] \times [j\varepsilon, (j+1)\varepsilon]. \quad (1)$$

We start with finding a square $s_{0,j}$ which contains the point $(1, 0)$, the intersection of the curve C_a with the x-axis, and then remove all other squares with $i = 0$. At $i = 1$, we take squares near the square $s_{0,j}$ containing $(1, 0)$ and check their feasibility.

Feasible squares are defined as follow:
(1) The square containing the point $(1, 0)$ is feasible.
(2) A square $s_{i,j}$ is feasible if there exists a feasible square $s_{i',j'}$ such that

(i) $i' < i$,

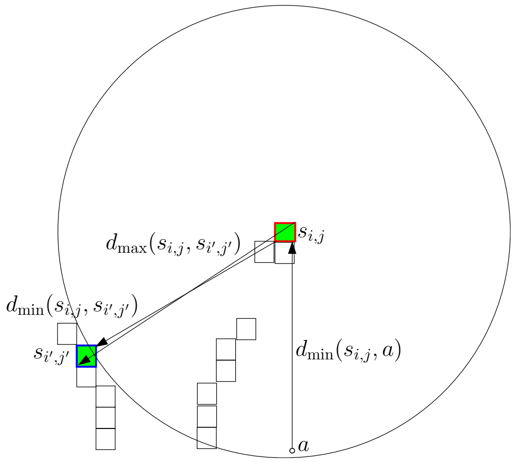

Figure 2: Feasible squares with related distances.

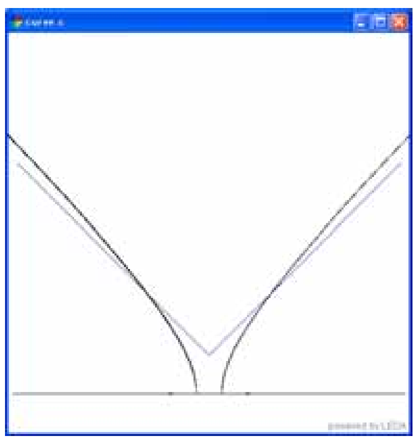

Figure 3: Two curves drawn by our approximation algorithm.

(ii) $[d_{\min}(s_{i,j}, a), d_{\max}(s_{i,j}, a)] \cap [d_{\min}(s_{i,j}, s_{i',-j'}), d_{\max}(s_{i,j}, s_{i',-j'})] \neq \emptyset$, and

(iii) there exists no square $s_{i'',j''}$ such that $d_{\max}(s_{i,j}, s_{i'',-j''}) < d_{\min}(s_{i,j}, a)$,

where note that if a square $s_{i,j}$ on the curve C_a is feasible then the square $s_{i,-j}$ is a feasible square on the curve C_b. $d_{\min}(s_{i,j}, a)$ and $d_{\max}(s_{i,j}, a)$ are the minimum and maximum distances between the square $s_{i,j}$ and the point a, and $d_{\min}(s, s')$ and $d_{\max}(s, s')$ are the minimum and maximum distances between two squares s and s'.

Once we have feasible squares $s_{i,j}, s_{i,j+1}, \ldots, s_{i,k}$ for i, a set of feasible squares for $i+1$ should start somewhere around j and end somewhere near k on $i+1$. Fig. 3 shows our implementation result.

4 More Characterizations

There are some curves and lines that characterize the equally-spaced curves C_a and C_b. Because of symmetricity of the curves we only consider the upper halves of C_a and C_b.

What is an optimal pair of half lines to approximate them to minimize the error? Because of their symmtricity, we can assume that a pair $y = cx + d, x \geq 0$ and $y = -cx + d, x \leq 0$ is optimal, where $c, d > 0$. An error for a point $p(x, y)$ on $y = cx + d$ is given by the difference between the distance from p to a and the length of a perpendicular line segment from p to the line $y = -cx + d$. If we denote the other endpoint of the segment by (x', y'), then the difference d is given by

$$\begin{aligned} d &= \sqrt{(x-1)^2 + y^2} - \sqrt{(x-x')^2 + (y-y')^2} \\ &= \frac{(x-1)^2 + y^2 - (x-x')^2 - (y-y')^2}{\sqrt{(x-1)^2 + y^2} + \sqrt{(x-x')^2 + (y-y')^2}}. \end{aligned}$$

For this difference to converge as x goes to infinity, the coefficient of x^2 in the numerator must be 0, that is, $c = 1$.

Now, the difference is simplified to

$$d = \frac{2(d-1)x + 1 + d^2}{\sqrt{(x-1)^2 + (x+d)^2} + \sqrt{2x^2}}.$$

It is minimized when $d = 1$. In fact, when $c = d = 1$, the difference converges to 0 as x goes to infinity.

We have shown that an optimal pair of half lines to approximate the curves is given by $y = x + 1, x \geq 0$ and $y = -x + 1, x \leq 0$. However, it does not imply that they are asymptotes of the curves. For any point p on C_a its corresponding point q_p on C_b that is closest to p on C_b is defined as the image of p. If we move a point p along C_a toward infinity, its image also moves on C_b in the direction away from the origin. Then, does it also go to infinity? The answer is no. Images cannot go beyond some point q_∞ on C_b.

This also means that if a point p on C_a is sufficiently far away from the origin then it must be close to the bisector of the two points a and q_∞. So, the bisectors can be considered as the asymptotes of our curves.

5 Future Direction

We are now working on a Voronoi diagram based on the curves defined here. We call it a Voronoi diagram with neutral zones. We have obtained preliminary results.

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