Optimal Area Triangulation with Angular Constraints

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Extended Abstract

1 Introduction

We study triangulations of planar sets of points. The problem of interest is to optimize the area of the individual triangles in the triangulation, or to find the MinMax and MaxMin area triangulations of the given point set. These two problems admit polynomial time algorithms in the case when the point set is a convex polygon. In the general case, the problems are of unknown complexity. The problem of finding the MinMax area triangulation is mentioned as a hard open problem in Edelsbrunner's book [2]. In this paper we discuss an approach for approximating these two optimal area triangulations with a triangulation that has angular restrictions imposed. This results in a subcubic algorithm for finding the approximating triangulation. The approximation ratio depends on angular parameters, analytical results on this are shown.

2 Angular restrictions and forbidden zones

Suppose the optimal area triangulation (either MaxMin or MinMax area) contains small angles, which is known to be unsuitable for practical purposes. Denote the smallest angle in the optimal triangulation (which we don't know) as β. We call triangulation that has all of its angles larger than β a β -triangulation. We want to construct an α-triangulation, which is "fatter" ($\alpha > \beta$), that approximates the optimal with a practical coefficient. Given the fact that all of the angles of the α-triangulation are going to be larger than α, we can define a region surrounding each edge of the triangulation, called forbidden zone. The forbidden zone of an edge is by definition a region that is empty of points of the original point set if the edge is in the triangulation. In these circumstances, the forbidden zone is a polygonal region, recursively defined by adding to the edge isosceles triangles with a base the edge itself, and base angles of α, and continuing this process outwards of the already tiled area infinitely. First four steps are shown in Figure 1. The parameters of the forbidden zone are fully determined by the length of the edge a and the angle α. The forbidden zone entirely contains a trapezoid with the given edge as a base, base angles of 3α and heighth of $(a/2)\tan\alpha$. The zone also entirely contains a circle surrounding each of the endpoints of the edge. Please refer to Figure 2 for an illustration.

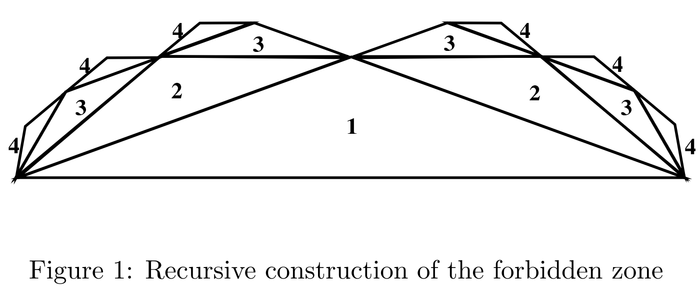

Figure 1: Recursive construction of the forbidden zone

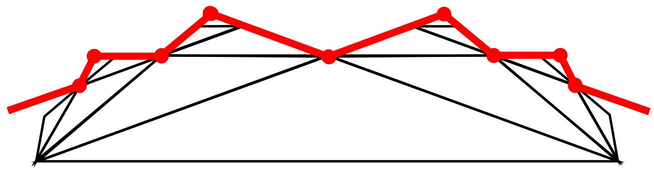

Figure 2: The border of the forbidden zone up to third order

3 Perfect matchings and bounds

To obtain the bounds for approximation factors, we use the perfect matching between the β -triangulation and the α-triangulation, as described in Aichholzer's paper [1]. We study all the possible cases of matched triangles and positions of the points with respect to the forbidden zones. Based on this we derive the following bounds of the approximation factors for the MaxMin area triangulation:

$$f_1 = max\left(\frac{1}{\tan\alpha\tan^2\beta\tan\frac{\beta}{2}}, \frac{1}{\sin\alpha\tan^2\beta}, \frac{2(1+k)(1+2k)}{\tan\alpha}, \frac{k^2}{\sin\alpha}\right)$$

and for the MinMax area triangulation:

$$f_2 = max\left(\frac{1}{\tan\beta\tan^2\alpha\tan\frac{\alpha}{2}}, \frac{1}{\sin\beta\tan^2\alpha}, \frac{1}{2(1-k)(1-2k)\sin\beta\tan\frac{\alpha}{2}}, \frac{1}{k_1^2\sin 2\beta}\right)$$

where k, k_1 and n are the following parameters of the forbidden zone:

$$k = \frac{\tan\alpha}{2\sin 3\alpha} \quad k_1 = \frac{1}{2^n\cos^n\alpha} \quad n = \left\lceil \frac{180^\circ}{2\alpha} - \frac{1}{2} \right\rceil$$

The approximation factor f_1 shows how many times the smallest area triangle in the approximating α -triangulation is smaller than the smallest area triangle in the optimal (MaxMin area) triangulation. Similarly, f_2 gives the ratio of the largest area triangle in the approximating triangulation, compared to the largest area triangle in the optimal (MinMax area) triangulation.

4 Algorithmic results and sample values

Based on the fact that we can compute the optimal 30°-triangulation (if it exists) by modified Klincsek's algorithm, or we can relax Delaunay by area equalizing flips, we achieve a subcubic algorithm that approximates the optimal area triangulations, by the above given factors. The value of α can be chosen from practical considerations. Here are some sample results, summarized in a table:

α	30	30	25	25	20	20	15
β	25	20	20	15	15	10	10
f_1	35.930	74.149	91.807	226.87	290.66	1010.1	1372.0
f_2	24.010	30.716	56.994	77.418	311.28	455.06	7900.1

Table 1: Sample values for f_1 and f_2

References

[1] O. Aichholzer, F. Aurenhammer, S.-W. Cheng, N. Katoh, G. Rote, M. Taschwer, and Y.-F. Xu. Triangulations intersect nicely. *Discrete and Computational Geometry*, 16(4):339–359, 1996.

[2] H. Edelsbrunner. *Geometry and Topology for Mesh Generation*. Cambridge University Press, UK, 2001.