# OrthoMap: Homeomorphism-guaranteeing normal-projection map between surfaces

Frédéric Chazal * André Lieutier † Jarek Rossignac ‡

## Extended abstract

In certain graphics applications, there is a need to establish a bijection between two surfaces for texture transfer (for instance in simplification) and for discrepancy measures. Furthermore, there is a desire to express one surface as the normal offset of another for compression, multi-resolution, detail preservation during editing. For this, given two surfaces $A$ and $B$, one associates to each point of $A$ a normal displacement distance (scalar field) to the corresponding point on $B$. The difficulty lies in the fact that in general, such mappings are difficult or impossible to establish and, when possible, often not very satisfactory. A natural correspondence would be to map each point p of $A$ onto its closest point $B.n(p)$ on $B$. The normal mapping $N(A, B)$ from $A$ onto $B$ associates with each point $p$ on $A$ its normal projection $B.n(p)$ on $B$. In general $N(A, B)$ is not a bijection (two different points $p$ and $q$ on $A$ may have the same images $B.n(p) = B.n(q)$) neither well defined (the closest point of a point $p$ on $A$ may not be uniquely defined). The set of points $p$ for which $B.n(p)$ is not unique is the medial axis $\mathcal{M}(B)$ of $B$ [3]. In this work, we develop a simple condition on $A$ and $B$ and prove that it guarantees that both $N(A, B)$ and $N(B, A)$ are bijective.

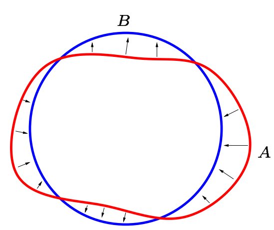


Figure 1: Two conformal curves $A$ and $B$

This condition involves the Hausdorff distance between $A$ and $B$ and the regularity of $A$ and $B$. The Hausdorff distance $H(A, B)$ between $A$ and $B$ may be defined in terms of $r$-thickening. The $r$-thickening $Gr(A)$ of $A$ is the union of all open balls of radius $r$ and center on $A$. Note that $Gr(A)$ is the Minkowski sum of $A$ with an open ball of radius $r$ and center at the origin. The $r$-thickening operator was used as a tool for offsetting, rounding and filleting operations [5] and for shape simplification [6]. The Hausdorff distance, $H(A, B)$, between two sets $A$ and $B$ is the smallest radius $r$ such that $A \subset Gr(B)$ and $B \subset Gr(A)$. A surface $A$ is $r$-regular if every point of it may be approached from both sides by an open ball of radius $r$ that is disjoint from $A$. More precisely, the $r$-thinning $Sr(M)$ of a set $M$ is the difference between $M$ and the union of open balls with center out of $M$ and the $r$-filleting $Fr(A)$ of $A$ is defined as $Sr(Gr(A))$. The surface $A$ is said to be $r$-regular if $Fr(A) = A$ [2]. Note that $Fr(A)$ contains all points that cannot be reached by a

*Université de Bourgogne, Institut de Mathématiques de Bourgogne, France
†Dassault Systèmes (Aix-en-Provence) and LMC/IMAG, Grenoble France
‡College of Computing, IRIS, GVU, Georgia Tech., Atlanta, Georgia

ball of radius $r$ whose interior does not interfere with $A$. The values $r$ for which $A$ is $r$-regular are related to the local feature size $\mathrm{lfs}(A)$ [1], and defined as the minimum distance between $A$ and its medial axis $\mathcal{M}(A)$. The surface $A$ is $r$-regular for all $r \leq \mathrm{lfs}(A)$.

**Definition:** *$A$ and $B$ are said to be conformal (to each other) when $A$ and $B$ are both $r$-regular for $r = H(A,B)/(2-\sqrt{2})$.*

The following theorem is the main result of this paper:

**Theorem:** *If surfaces $A$ and $B$ are conformal, then the normal mapping $N(A,B)$ is bijective.*

Moreover, $N(A,B)$ allows to define an explicit isotopy (i.e. a continuous deformation of $A$ into $B$) between $A$ and $B$ (see [4] for a precise definition). The proof of the theorem is cast in precise mathematical formalism which allows to prove this theorem for manifolds in any dimension. We also show that conformality is a tight condition by giving an example of two curves $A$ and $B$ that are $r$-regular for any $r \leq H(A,B)/(2-\sqrt{2})$, but not conformal, and hence $N(A,B)$ is not bijective (see figure 2).

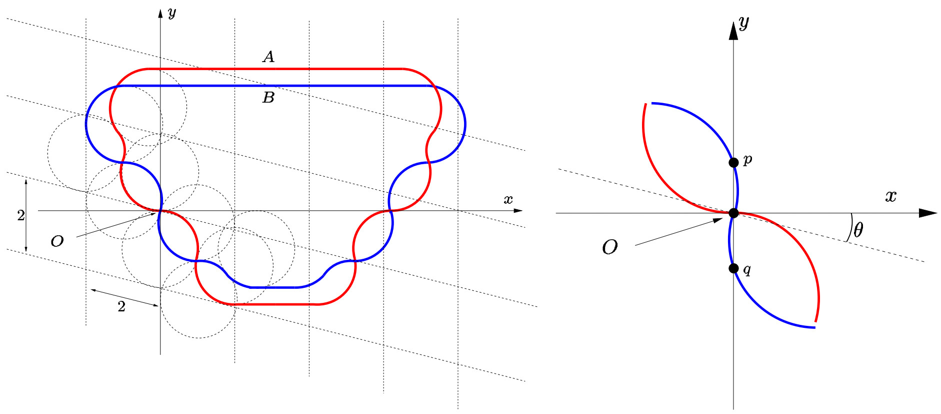


Figure 2: Two curves $A$ and $B$ showing that optimality is tight. Right part is a zoom of a neigborhood of 0 : $A.n(p) = A.n(q)$

In summary, we have provided a sufficient and tight condition to ensure that the normal mapping between two smooth $(n-1)$-dimensional manifolds in $n$-D is a bijection. The condition links the minimum regularity of the manifolds to their Hausdorff distance.

# References


[1] N. Amenta and M. Bern *Surface Reconstruction by Voronoi Filtering* Discrete and Computational Geometry, no 22 pp.481-504(1999)

[2] D. Attali, *r-Regular shape reconstruction from unorganized points*, Computational Geometry (1997) 248-253.

[3] H. Blum, *A transformation for extracting new descriptors of shape*, In W. Wathen-Dunn, editor, Models for the Perception of Speech and Visual Form, p.362-380, MIT Press, 1967.

[4] F. Chazal, D. Cohen-Steiner, *A condition for isotopic approximation,* proc. ACM Symp. Solid Modeling and Applications 2004.

[5] J. Rossignac, A. Requicha, *Offsetting Operations in Solid Modelling*, Computer-Aided Geometric Design, Vol. 3, pp. 129-148, 1986.

[6] J. Williams, J. Rossignac, *Mason: Morphological Simplification* GVU Tech. Report GIT-GVU-04-05 (Mason2.pdf) available from http://www.gvu.gatech.edu/ jarek/papers.html