Unfolding Smooth Primsatoids

ABSTRACT

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Abstract

We define a notion for unfolding smooth, ruled surfaces, and prove that every smooth prismatoid (the convex hull of two smooth curves lying in parallel planes), has a nonoverlapping "volcano unfolding." These unfoldings keep the base intact, unfold the sides outward, splayed around the base, and attach the top to the tip of some side rib. Our result answers a question for smooth prismatoids whose analog for polyhedral prismatoids remains unsolved.

Introduction. It is a long-unsolved problem to determine whether or not every convex polyhedron can be cut along its edges and unfolded flat into the plane to a single nonoverlapping simple polygon (see, e.g., [O'R00]), the *net*. These unfoldings are known as *edge unfoldings* because the surface cuts are along edges. In this paper,[1] we generalize edge unfoldings to certain piecewise-smooth ruled surfaces,and show that smooth prismatoids can always be unfolded without overlap. Our hope is that the smooth case will inform the polyhedral case.

Pyramids and Cones. A *pryamid* is a polyhedron that is the convex hull of a convex *base* polygon B and an *apex* v above the plane containing the base. The *side faces* are all triangles. It is trivial to unfold a pyramid without overlap: cut all side edges and no base edge. This produces what might be called a *volcano* unfolding. Examples are shown in Fig. 1(a,b) for regular polygon bases.

We generalize pyramids to *cones*: shapes that are the convex hull of a smooth convex curve base B lying in the xy-plane, and a point apex v above the plane. We define the volcano unfolding of a cone to be the natural limiting shape as the number of vertices of base polygonal approximations goes to infinity, and each side triangle approaches a segment *rib*. This limiting process is illustrated in Fig. 1(c). For any point $b \in \partial B$, the segment vb is unfolded across the tangent to B at b. Note that

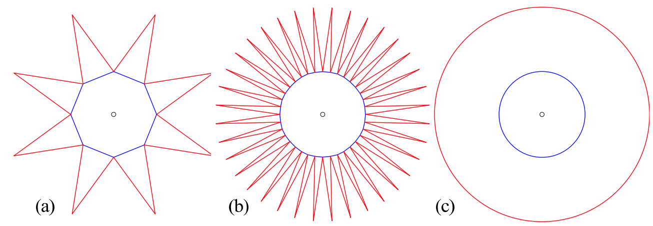

Figure 1: Unfoldings of regular pyramids (a-b) approaching the unfolding of a cone (c).

this net for a cone is no longer an unfolding that could be produced by paper, because the area increases.

Main Result. Our main result concerns a shape known as a *prismatoid*, the convex hull of two convex polygons A and B lying in parallel planes. There is no algorithm for edge-unfolding prismatoids. Our concentration in this paper is on *smooth prismatoids*, which we define as the convex hull of two smooth convex curves A above and B below, lying in parallel planes. A volcano unfolding of a smooth prismatoid unfolds every rib segment ab of the convex hull, $a \in \partial A$ and $b \in \partial B$, across the tangent to B at b, into the xy-plane, surrounding the base B, with the top A attached to one appropriately chosen rib. The main result of this paper is that every smooth prismatoid has a nonoverlapping volcano unfolding. Fig. 2 illustrates the side unfolding of a prismatoid; the top A must be carefully placed tangent to the side unfolding and on the convex hull of that unfolding.

References

[DO04] Erik D. Demaine and Joseph O'Rourke. *Folding and Unfolding in Computational Geometry.* 2004. Monograph in preparation.

[O'R00] Joseph O'Rourke. Folding and unfolding in computational geometry. In *Discrete Comput. Geom.*, volume 1763 of *Lecture Notes Comput. Sci.*, pages 258–266. Springer-Verlag, 2000. Papers from the *Japan Conf. Discrete Comput. Geom.*, Tokyo, Dec. 1998.

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[1] See http://arxiv.org/abs/cs/0407063 for the full version.

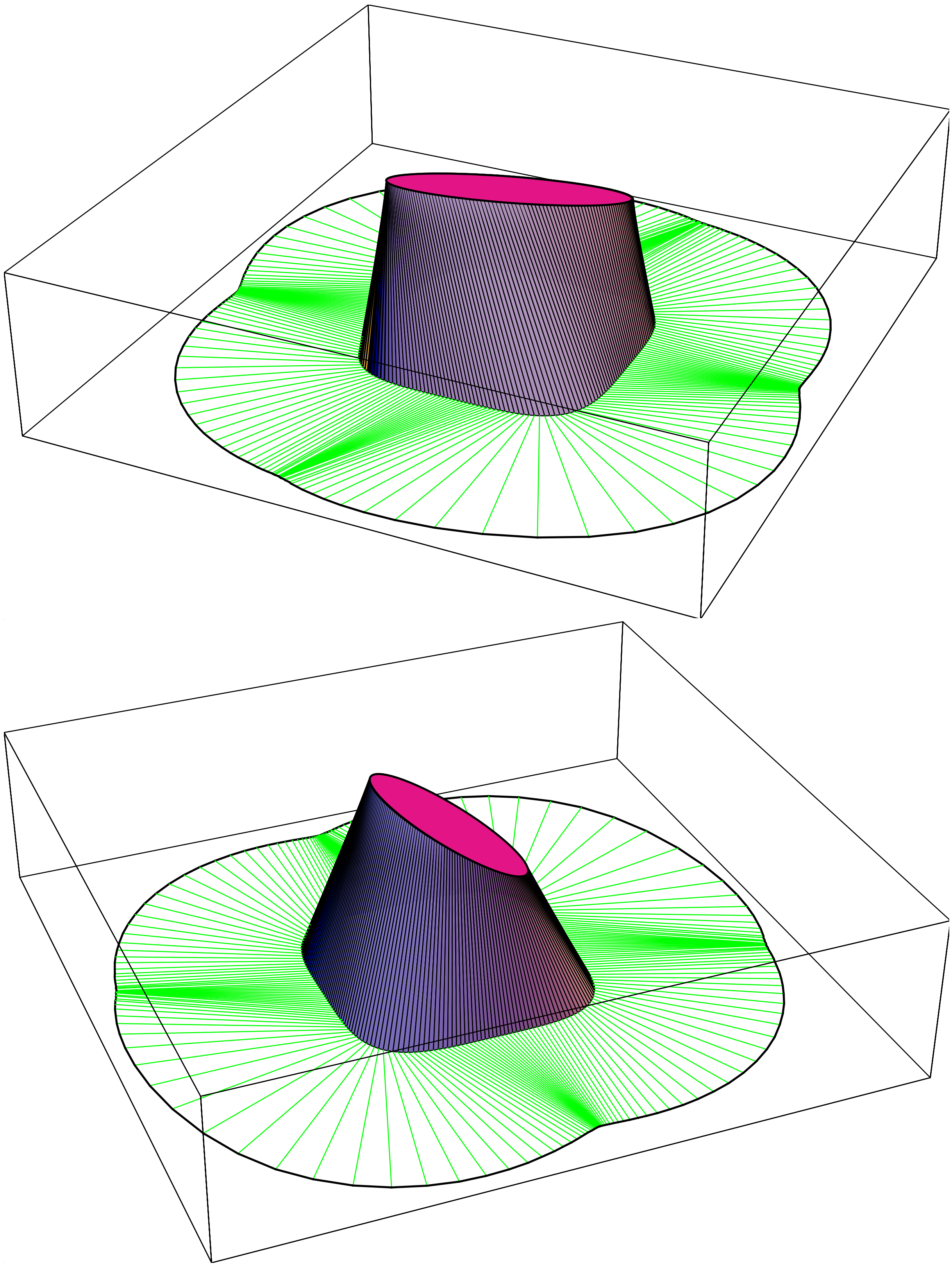

Figure 2: Two views of the side unfolding of a 3D prismatoid. The top A is an ellipse in a plane parallel to the base.